Saturation Heuristic for Faster Bisimulation with Petri Nets

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Project Presentation for Oral Qualifying Examination

Outline

Abstract

The present work applies the *Saturation* heuristic and interleaved MDD partition representation to the bisimulation problem. For systems with deterministic transition relations (Petri Nets) bisimulation can be expressed as a state-space exploration problem, for which the saturation heuristic has been found to be quite efficient. The present work compares our novel saturation-based bisimulation algorithm with other fully-implicit and partially-implicit methods (using non-interleaved MDDs) in the context of the SMART verification tool. We found that with some models having very many equivalence classes in their bisimulation partitions, our novel algorithm gave much better speed performance than any of the other algorithms tested. With other models, our novel algorithm performed only slightly less well than the fastest tested algorithm.

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Definition of Bisimulation

$\mathcal{B}$ is a bisimulation of colored, labeled FSA: $\langle S, C, T\rangle \mid$
$C \in S \rightarrow \mathit{color} \wedge T \subseteq S \times \mathit{label} \times S$, iff:
$\mathcal{B} \subseteq S \times S \wedge \forall\langle s_1, s_2\rangle \in \mathcal{B} : [\ C(s_1) = C(s_2) \wedge (\forall\langle s, l, s_1'\rangle \in T :$
$s = s_1 \implies \exists s_2' \in S : T(s_2, l, s_2') \wedge \mathcal{B}(s_1', s_2')) \wedge (\forall\langle s, l, s_2'\rangle \in T :$
$s = s_2 \implies \exists s_1' \in S : T(s_1, l, s_1') \wedge \mathcal{B}(s_1', s_2'))\ ]$

Original Definition of Bisimulation (Milner 1989)

4.2 Strong bisimulation

The above discussion leads us to consider an equivalence relation with the following property:

> P and Q are equivalent iff, for every action α, every α-derivative of P is equivalent to some α-derivative of Q, and conversely.

...

Definition 1 A binary relation $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ over agents is a *strong bisimulation* if $(P, Q) \in \mathcal{S}$ implies, for all $\alpha \in Act$,

(i) Whenever $P \xrightarrow{\alpha} P'$ then, for some Q', $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{S}$
(ii) Whenever $Q \xrightarrow{\alpha} Q'$ then, for some P', $P \xrightarrow{\alpha} P'$ and $(P', Q') \in \mathcal{S}$ ■

Why Bisimulation?

Bisimulation is . . .

- A special case of Lumping
  (A minimization problem for Markov systems) to simplify subsequent numeric computations
- An extensional notion of equivalence of states (FSA)

Notation:

- $R \subseteq S \times S$ — A relation between states
- $\mathcal{B}(s_1, s_2)$ or $\langle s_1, s_2 \rangle \in \mathcal{B}$ — "s_1 and s_2 are bisimilar"
- "$\sim$" — The Largest Bisimulation

A Bisimulation is ...

Definition

- (Given a colored, labeled transition system,(st,col,tran) $\langle S, C, T\rangle \mid C \in S \rightarrow \mathit{color} \wedge T \subseteq S \times \mathit{label} \times S$,
- A Bisimulation $\mathcal{B}$ is a 2-ary relation on S where: $\mathcal{B} \subseteq S \times S \wedge$
- Each pair in $\mathcal{B}$ has the same color, $\forall\langle s_1, s_2\rangle \in \mathcal{B} : C(s_1) = C(s_2) \wedge$
- And has matching transitions to pairs in $\mathcal{B}$ $\forall\langle s, l, s_1'\rangle \in T : s = s_1 \implies \exists s_2' \in S : T(s_2, l, s_2') \wedge \mathcal{B}(s_1', s_2')$

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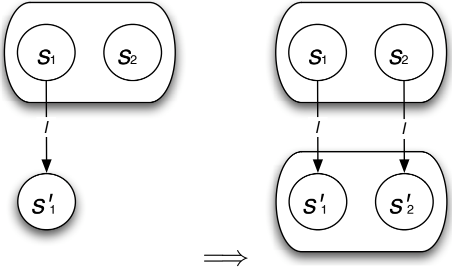
Matching Transitions to Pairs in $\mathcal{B}$.

$$\forall \langle s_1, s_2 \rangle \in \mathcal{B} :$$
$$\forall \langle s, l, s_1' \rangle \in T : s = s_1 \implies \exists s_2' \in S : T(s_2, l, s_2') \wedge \mathcal{B}(s_1', s_2')$$



The (Largest) Bisimulation is ...

Definition

The Largest Bisimulation, "$\sim$" is the union of all bisimulations $\mathcal{B}$

And is an equivalence relation.

Original Definition of "$\sim$" (Milner 1989)

Definition 2 P and Q are *strongly equivalent* or *strongly bisimilar*, written $P \sim Q$, if $(P, Q) \in \mathcal{S}$ for some strong bisimulation $\mathcal{S}$. This may be equivalently expressed as follows:

$$\sim \;=\; \bigcup\{\mathcal{S} \;:\; \mathcal{S} \text{ is a strong bisimulation}\} \qquad \blacksquare$$

Proposition 2

(1) $\sim$ is the largest strong bisimulation.
(2) $\sim$ is an equivalence relation.

Relational Coarsest Partition = Largest Bisimulation.

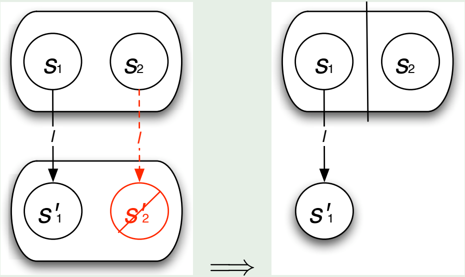
Generic iterative *splitting* algorithm:

- Iterative update of some equivalence relation variable R.
- Start with $R =$ coarsest partition of state space S, $S \times S$ ($\sim \subseteq R$)
- Initially *split* R based on state color
- Iteratively remove implausible members from R when required by definition of Bisimulation, by splitting R into smaller blocks B_*.
  $\forall \langle s, l, s_1' \rangle \in T : s = s_1 \implies \exists s_2' \in S : T(s_2, l, s_2') \wedge R(s_1', s_2')$
- Iteration continues until all blocks have been used as splitters (inherited stability, block unions).
- May iterate over transition labels. Algorithm cores are often described without reference to labeling.

Splitting.

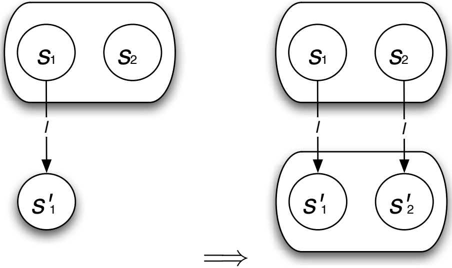
$$\forall \langle s_1, s_2 \rangle \in R :$$
$$\forall \langle s, l, s_1' \rangle \in T : s = s_1 \implies \exists s_2' \in S : T(s_2, l, s_2') \wedge R(s_1', s_2')$$

Example



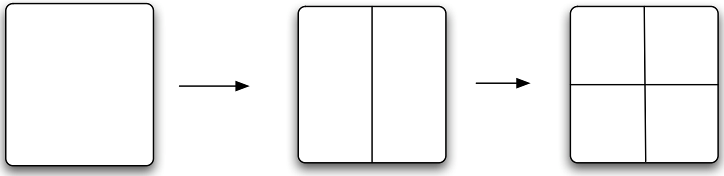
Matching Transitions to Pairs in R.

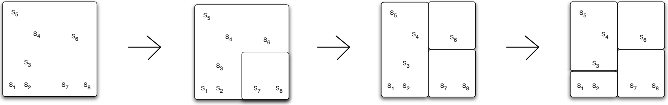
$$\forall \langle s_1, s_2 \rangle \in R :$$
$$\forall \langle s, l, s_1' \rangle \in T : s = s_1 \implies \exists s_2' \in S : T(s_2, l, s_2') \wedge R(s_1', s_2')$$



Outline

Splitting produces hierarchy of partition blocks

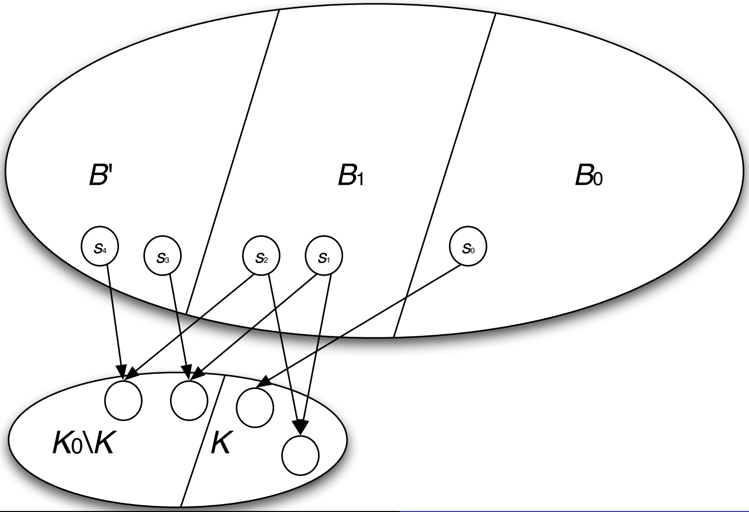




"Process The Smaller Half." $O(m \log n)$

- Start with $R =$ coarsest partition of state space S, $S \times S$
- First split uses S as splitter. Separates states with no transitions.
- Remember hierarchy of split blocks for use as splitters
- Use 2 splitters K and $K_0 \setminus K$, where K_0 was already a splitter.
- Iteratively split blocks B into smaller blocks B_0, B_1, and B'
- Maintain reverse adjacency lists
- Maintain counts of edges from states to states in splitter blocks

"Process The Smaller Half." $O(m \log n)$

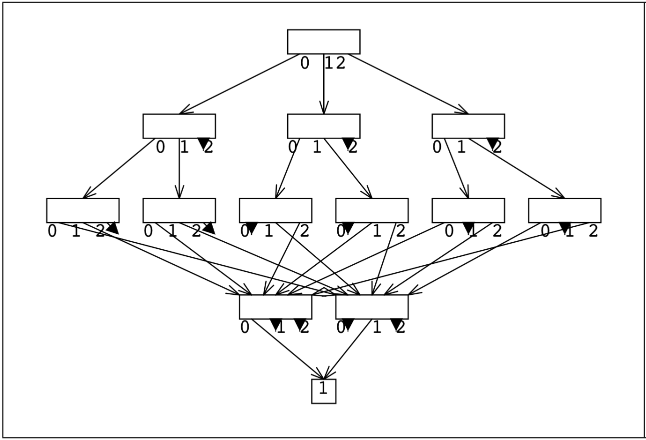


"Process The Smaller Half." $O(m \log n)$

- Uses edge counts to distinguish between members of B_0 and B_1.
- Avoids processing members of B' and $K_0 \setminus K$ (by reusing structures).
- Update edge counts.
- Each state s occurs in at most $\log n$ splitters.
- Each edge participates in at most $O(\log n)$ splitting operations
- $T = O(m \log n)$

Outline

"Symbolic Methods" $\neq$ Mathematica ®$_{(WolframResearch)}$



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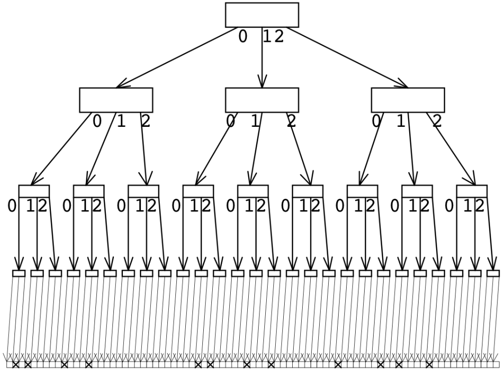
Multi-Way Decision Diagrams Represent Relations

- Each path in MDD (graph) corresponds to tuple in relation.
- Canonical: sharing $\leftrightarrow$ compression, comparison, *unique* table, non-mutable.
- Efficient memoized recursive algorithms for set operations: ($\in$ (not memoized), $|()|$, $\cup$, $\cap$, $\setminus$, $\subseteq$).
- Efficient memoized recursive algorithms for functional operations: ($\circ$, $\exists$, $\forall$).
- Set operations implemented in SMART MDD library.
- SMART Saturation algorithm for transitive closure (state space exploration).
- "Quasi-reduced", with "NULL" edges
- Variable ordering matters.

Set = Boolean Table ($\hat{S} = [1,3]^4$)



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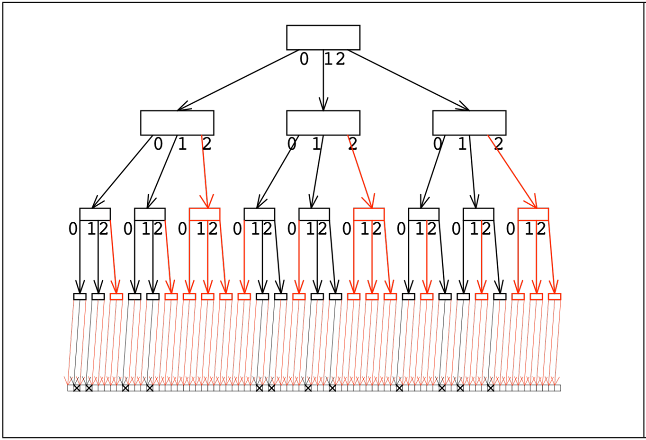
Empty Subsets



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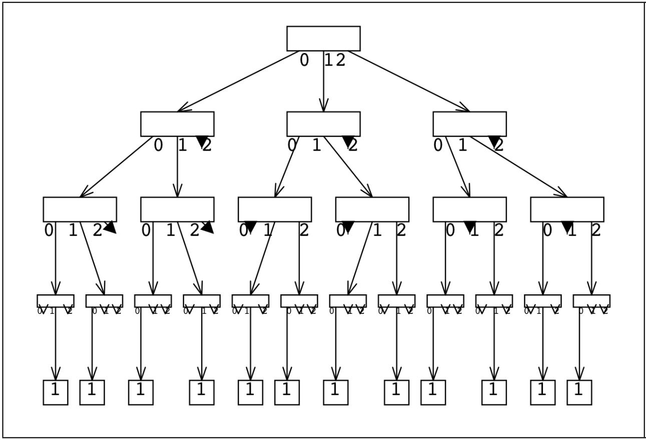
Replace with “NULL” Edges



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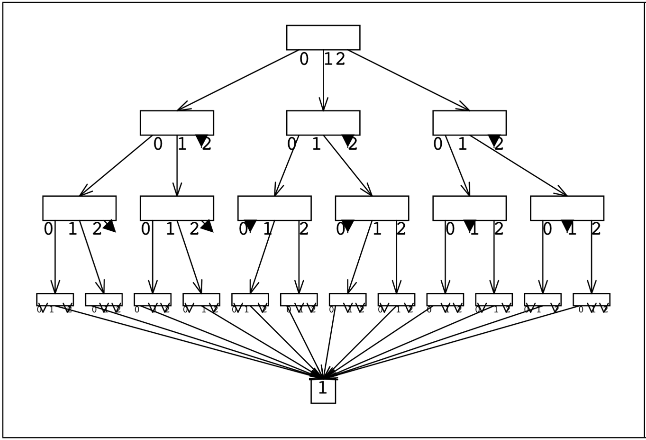
Quasi-Reduce at Leaf Level



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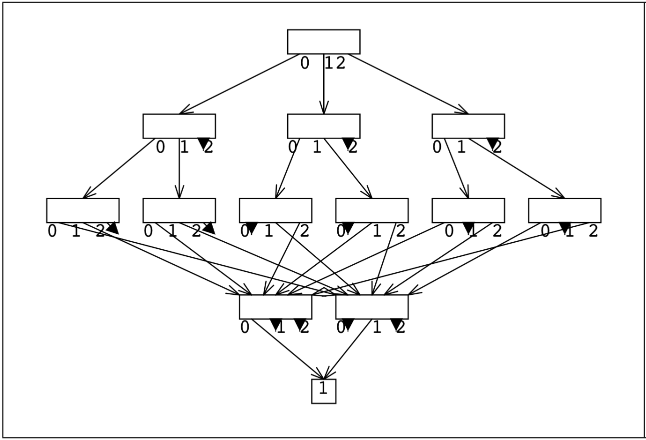
Quasi-Reduced MDD



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Memoized Recursive Algorithm for Set Difference ("\")

Algorithm $\mathcal{R} \leftarrow \mathcal{X} \setminus \mathcal{Y}$

1. Handle a few special cases before checking cache:
   1. If $\mathcal{X} = \emptyset$ then return with $\mathcal{R} \leftarrow \emptyset$
   2. If $\mathcal{Y} = \emptyset$ then return with $\mathcal{R} \leftarrow \mathcal{X}$
   3. If $\mathcal{X} = \mathcal{Y}$ then return with $\mathcal{R} \leftarrow \emptyset$
2. If the cache has $\mathcal{X} \setminus \mathcal{Y}$ then return with $\mathcal{R} \leftarrow$ cached value
3. Construct new MDD node $\mathcal{R}$ as follows:
4. Recursively call: $\mathcal{R}_i \leftarrow \mathcal{X}_i \setminus \mathcal{Y}_i$, for each variable value i
5. If $\forall i : \mathcal{R}_i = \emptyset$ then $\mathcal{R} \leftarrow \emptyset$
6. Make $\mathcal{R}$ canonical: $\mathcal{R} \leftarrow unique(\mathcal{R})$
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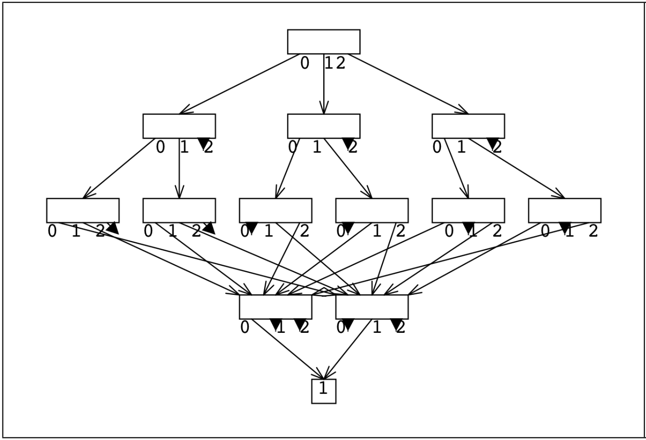
Variable Ordering Matters (1)



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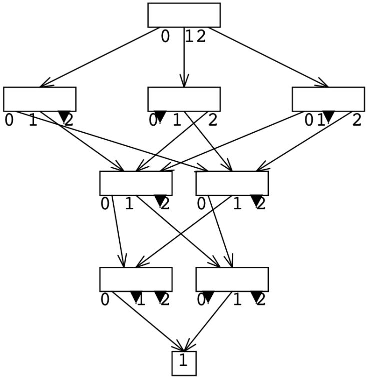
Variable Ordering Matters (2)



“0001”	“0001”
“0100”	“0010”
“0010”	“0100”
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“1101”	“1011”
“1200”	“1020”
“1110”	“1110”
“1211”	“1121”
“2000”	“2000”
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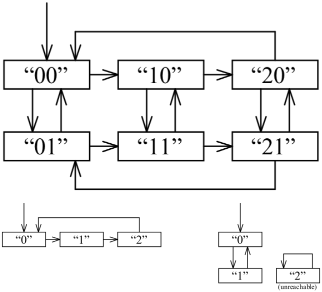
Represent FSAs as Relations (and MDDs)



⟨ "00" → "01" ⟩	"0001"
⟨ "00" → "10" ⟩	"0010"
⟨ "01" → "00" ⟩	"0100"
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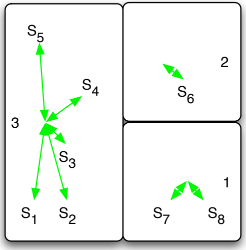
Represent FSAs as Relations (and MDDs)

- Each state variable corresponds to a (set of) variables in tuple.
- Each transition in FSA corresponds to tuple in transition relation.
- Interleaved ordering of variables of source and destination states of transition relation usually yields relatively compact MDDs.
- SMART$_2$ produces MDDs of transition relations in interleaved form.

Alternate Ways to Represent Partitions as Relations

1. Equivalence Relation: $\langle s_1, s_2 \rangle \mid s_1, s_2 \in S$
2. List of Partition Blocks $B_1, B_2, B_3, B_4, \ldots \mid B_* \subseteq S$
3. Block Numbering $\langle s, n \rangle \mid s \in S, n \in \mathbb{N}$



1

Alternate Ways to Represent Partitions as Relations

1. Equivalence Relation: $\langle s_1, s_2 \rangle \mid s_1, s_2 \in S$
2. List of Partition Blocks $B_1, B_2, B_3, B_4, \ldots \mid B_* \subseteq S$
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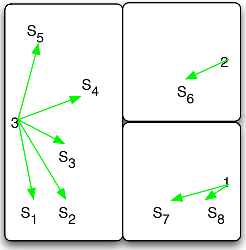
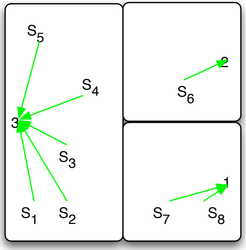


2

Alternate Ways to Represent Partitions as Relations

1. Equivalence Relation:
   $\langle s_1, s_2 \rangle \mid s_1, s_2 \in S$
2. List of Partition Blocks
   $B_1, B_2, B_3, B_4, \ldots \mid$
   $B_* \subseteq S$
3. Block Numbering
   $\langle s, n \rangle \mid s \in S, n \in \mathbb{N}$



Ways to Represent Partitions as MDDs

1. Equivalence Relation (Non-Interleaved) (link)
   - $\langle s_1, s_2 \rangle \mid s_1, s_2 \in S$
   - Variable ordering: $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$
2. Equivalence Relation (Interleaved) (link)
   - $\langle s_1, s_2 \rangle \mid s_1, s_2 \in S$
   - Variable ordering: $x_1, y_1, x_2, y_2, x_3, y_3, \ldots$
3. Lists of Partition Blocks (link)
   - $B_1, B_2, B_3, B_4, \ldots \mid B_* \subseteq S$
   - Variable ordering: $x_1, x_2, x_3, \ldots$
4. Block Numbering/function of state (link)
   - $\langle s, n \rangle \mid s \in S, n \in \mathbb{N}$
   - Variable ordering: $x_1, x_2, x_3, \ldots k_1, k_2, k_3, \ldots$

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Outline

1. Overview
   - Abstract
   - Bisimulation
2. Algorithms for Bisimulation
   - Paige and Tarjan
   - Symbolic Methods
   - Previous Work
3. Our Work
   - Our Algorithms (fully implicit Algorithm 1)
   - Our Algorithms (Hybrid Algorithm H)
   - Our Algorithms (Saturation Algorithm A)
4. Results and Future Work

Generic Signature-Based Splitting Algorithm

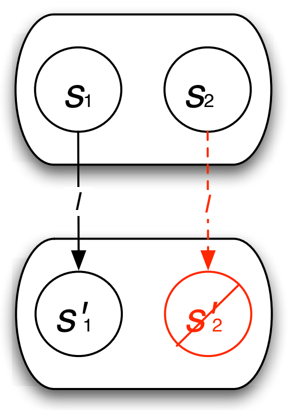
- Split each partition block using all blocks as splitters.
- State Space: S, Partition: $P \in S \rightarrow Block$, Transition: $Q \subseteq S \times S$, Signature: T
- Signature of a state s includes set of partition blocks to which s has transitions.
- Signature includes current partition block where s resides.
- Signature often described without edge labeling.
- Define new partition of S, with a block for each signature.

Algorithm: Signature-Based Splitting

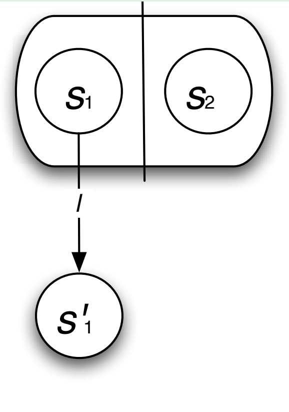
1. Signature: $T(s) = \langle$ P(s), $\{P(s')|\langle s, s'\rangle \in Q\}$ $\rangle$.
2. New Partition: $P'(s) = f(T(s))$ (for some bijection f)
3. Repeat 1;2;$P \leftarrow P'$ until $P = P'$

Splitting.

Example

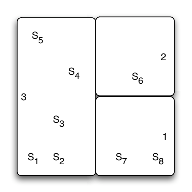


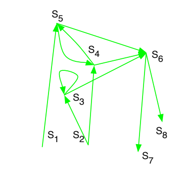
$\Longrightarrow$

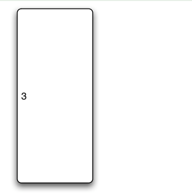


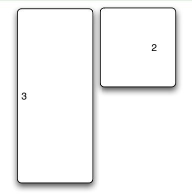
Generic Signature-Based Splitting Algorithm

Example

partition (P): 

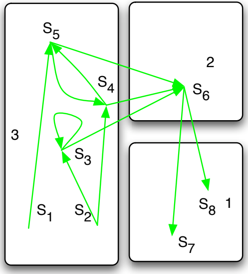
transitions (Q): 

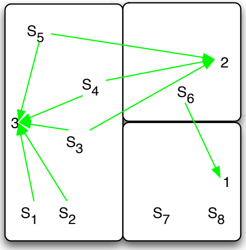
$T(\ s_4\ ) = \langle$ 

$, \{$ 

$\}\rangle$

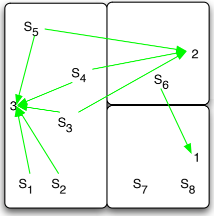
Generic Signature-Based Splitting Algorithm

P, Q: 

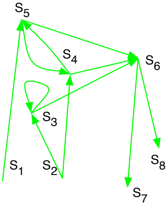
, T: 

Generic Signature-Based Splitting Algorithm

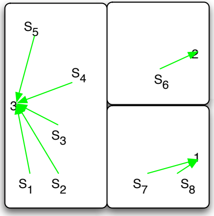
$T = Q \circ P$



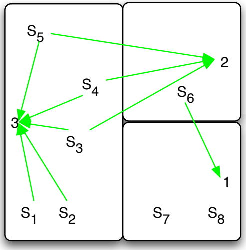
$=$

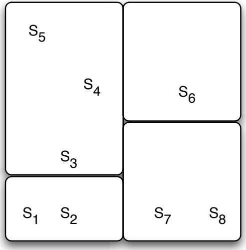


$\circ$



Generic Signature-Based Splitting Algorithm

T: 

, P': 

Generic Signature-Based Splitting Algorithm

- Split each partition block using all blocks as splitters.
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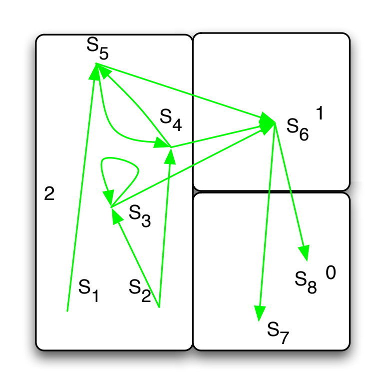
2. New Partition: $P'(s) = f(T(s))$ (for some bijection f)
3. Repeat 1;2;$P \leftarrow P'$ until $P = P'$

Algorithm: Rank-Based Initial Partition

- Agostino Dovier, Carla Piazza, and Alberto Policriti (2004).
- Linear symbolic steps.
- Produces *rank*-based partition
- Partition representation: lists of partition blocks
- Needs other block splitting algorithm to finish.
- Apply other algorithm to blocks in rank order.
- Strongly connected components cause problems.
- Extract rank-1 elements: $R_1 \leftarrow S \setminus \mathit{preimage}(S)$

Algorithm: Rank-Based Initial Partition

Example



Algorithm: Forwarding, Splitting, Ordering

- Ralf Wimmer, Marc Herbstritt, and Bernd Becker (2007).
- Partition representation: lists of blocks AND numbering function
- Algorithm maintains signature and partition.
- Forwarding: Immediately update partition numbering function.
- Split-drive refinement: Only attempt splitting on blocks that might be split.
- Block ordering: Split blocks that might propagate splitting most.

History

1. Our previous work (summary)
   - Review lumping algorithms.
   - Ideas: Interleaved partition representation, depth-based
   - Limit scope to bisimulation instead of lumping.
   - Algorithm 1: Relational interleaved partition refinement
   - Implement interleaved partition refinement for bisimulation.
   - Review bisimulation: Bouali and De Simone (1992).
   - Implement hybrid algorithm to compare representations
   - Hybrid algorithm was usually faster, for models we used
2. Attempted improvements
   - Increased integration of set operations (minor variations)
   - Calculate bisimulation over $\hat{S}$ (often much worse)
   - Symbolic block numbering in Hybrid algorithm (couldn't)
   - Idea: Saturation construction of $\approx$

Outline

Symbolic Bisimulation Minimization

- Amar Bouali and Robert De Simone (1992).
- Partition representation: Equivalence relation (interleaved or non-interleaved)
- Transition representation: Relation (interleaved or non-interleaved (respectively))
- Similar to generic signature-based splitting algorithm.

Our Implementation of Bouali and De Simone's Algorithm

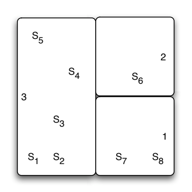
- Partition representation: Equivalence relation (interleaved)
- Transition representation: Relation (interleaved)
- Similar to generic signature-based splitting algorithm, except:
- Equivalence relation allows signature without current partition number.

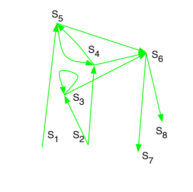
Algorithm 1 Signature Formula

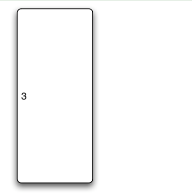
- S — State space
- $E \subseteq S \times S$ — Equivalence relation
- $Q_{(t)} \subseteq \hat{S} \times \hat{S}$ — Transition relation (for transition t)
- $T \subseteq S \times S = Q \circ E$ — Signatures
- $T(s_1, s_3)$ iff $\exists s_2 \in S : Q(s_1, s_2) \wedge E(s_2, s_3) \wedge S(s_1)$

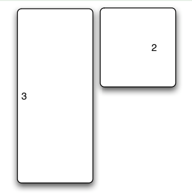
Generic Signature-Based Splitting Algorithm

Example

partition (P): 

transitions (Q): 

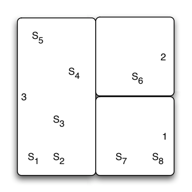
$T(\ s_4\ ) = \langle$ 

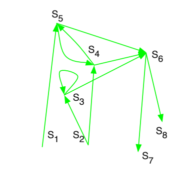
$, \{$ 

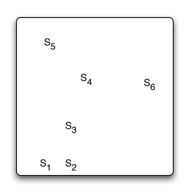
$\}\rangle$

Algorithm 1 Signature

Example

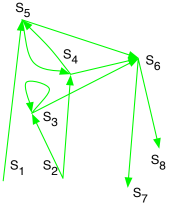
partition (P): 

transitions (Q): 

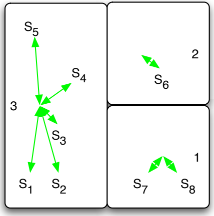
$T(\ s_4\ ) =$ 

Algorithm 1 Signature

$T = Q \circ P$



$=$ $\circ$



Algorithm 1 Signature Calculation

- State space MDD: $\mathcal{S}$
- Interleaved equivalence relation MDD: $\mathcal{E} \subseteq S \times S$
- Interleaved transition relation MDD: $\mathcal{Q} \subseteq \hat{S} \times \hat{S}$
- Signatures MDD: $\mathcal{T} \leftarrow proj_{\vee 3}((DC_2(\mathcal{Q}, \mathcal{S})) \cap (DC_1(\mathcal{E}, \mathcal{S})))$
- $T(s_1, s_3)$ iff $\exists s_2 \in S : Q(s_1, s_2) \wedge E(s_2, s_3) \wedge S(s_1)$

Definitions for Extra Operators

- $DC_1(\mathcal{E},\mathcal{S}) \triangleq \underline{\mathcal{E}}$, where $\underline{\mathcal{E}}(x,y,z) = \mathcal{E}(y,z) \wedge \mathcal{S}(x)$
- $DC_2(\mathcal{Q},\mathcal{S}) \triangleq \underline{\mathcal{Q}}$, where $\underline{\mathcal{Q}}(x,y,z) = \mathcal{Q}(x,z) \wedge \mathcal{S}(y)$
- $proj_{\vee 3}(\mathcal{F}) \triangleq \mathcal{F}'$, where $\mathcal{F}'(x,y) = \bigvee c : \mathcal{F}(x,y,c)$
- Signatures MDD: $\mathcal{T} \leftarrow proj_{\vee 3}((DC_2(\mathcal{Q},\mathcal{S})) \cap (DC_1(\mathcal{E},\mathcal{S})))$
- $\mathcal{T}(x,y) \leftarrow \bigvee z : (\underline{\mathcal{Q}}(x,(y),z) \wedge \underline{\mathcal{E}}((x),y,z))$
- $\mathcal{T}(x,y) \leftarrow \bigvee z : (\mathcal{Q}(x,z) \wedge \mathcal{S}(y) \wedge \mathcal{E}(y,z) \wedge \mathcal{S}(x))$
- $\mathcal{T}(s_1,s_3) \leftarrow \bigvee s_2 : (\mathcal{Q}(s_1,s_2) \wedge \mathcal{S}(s_3) \wedge \mathcal{E}(s_3,s_2) \wedge \mathcal{S}(s_1))$
- $\mathcal{T}(s_1,s_3) \leftarrow \bigvee s_2 : (\mathcal{Q}(s_1,s_2) \wedge \mathcal{E}(s_3,s_2) \wedge \mathcal{S}(s_1))$
- $T(s_1,s_3)$ iff $\exists s_2 \in S : Q(s_1,s_2) \wedge E(s_2,s_3) \wedge S(s_1)$

Algorithm 1 Equivalence Relation Formula

- S — State space
- $T \subseteq S \times S = Q \circ E$ — Signatures
- $\Delta E \subseteq S \times S$ — Equivalence relation update
- $\Delta E(s_1, s_3)$ iff $\forall s_2 \in S : T(s_1, s_2) = T(s_3, s_2)$
- $E' \leftarrow E \wedge \Delta E$

Algorithm 1 Equivalence Relation Calculation

- State space MDD: $\mathcal{S}$
- Signatures MDD: $\mathcal{T}$
- $\Delta\mathcal{E} \leftarrow proj_{\wedge 3}(DC_2(\mathcal{T},\mathcal{S}) \equiv DC_1(\mathcal{T},\mathcal{S}))$
- $\mathcal{E}' \leftarrow \mathcal{E} \wedge \Delta\mathcal{E}$
- $\Delta E(s_1, s_3)$ iff $\forall s_2 \in S : T(s_1, s_2) = T(s_3, s_2)$

Algorithm 1 Equivalence Relation Calculation

- $\Delta\mathcal{E} \leftarrow proj_{\wedge 3}(DC_2(\mathcal{T}, \mathcal{S}) \equiv DC_1(\mathcal{T}, \mathcal{S}))$
- $\mathcal{E}' \leftarrow \mathcal{E} \wedge \Delta\mathcal{E}$
- $\overline{\Delta\mathcal{E}} \leftarrow proj_{\vee 3}(DC_2(\mathcal{T}, \mathcal{S}) \dot\cup DC_1(\mathcal{T}, \mathcal{S}))$
- where $x \dot\cup y \triangleq (x \setminus y) \cup (y \setminus x)$
- $\mathcal{E}' \leftarrow \mathcal{E} \setminus \overline{\Delta\mathcal{E}}$

Algorithm 1

Given: Initial partition in variable $\mathcal{E}$, transition relation in $\mathcal{Q}$, state space in $\mathcal{S}$.
Returns final partition in $\mathcal{E}$.

Algorithm: refinement of equivalence relation using signature relation

Repeat:

- $\mathcal{E}_{old} \leftarrow \mathcal{E}$
- $\mathcal{T} \leftarrow proj_{\vee 3}((DC_2(\mathcal{Q},\mathcal{S})) \cap (DC_1(\mathcal{E},\mathcal{S})))$
- $\overline{\Delta\mathcal{E}} \leftarrow proj_{\vee 3}(DC_2(\mathcal{T},\mathcal{S}) \cup DC_1(\mathcal{T},\mathcal{S}))$
- $\mathcal{E} \leftarrow \mathcal{E} \setminus \overline{\Delta\mathcal{E}}$

Until $\mathcal{E} = \mathcal{E}_{old}$

Algorithm 1 with Transition Labeling

Given: Initial partition in variable $\mathcal{E}$, transition relation in $\mathcal{Q}$, state space in $\mathcal{S}$.
Returns final partition in $\mathcal{E}$.

Algorithm: refinement of equivalence relation using signature relation

Repeat:

- $\mathcal{E}_{old} \leftarrow \mathcal{E}$
- For each $t \in label$ loop:
  - $\mathcal{T} \leftarrow proj_{\vee 3}((DC_2(\mathcal{Q}_t, \mathcal{S})) \cap (DC_1(\mathcal{E}, \mathcal{S})))$
  - $\overline{\Delta\mathcal{E}} \leftarrow proj_{\vee 3}(DC_2(\mathcal{T}, \mathcal{S}) \cup DC_1(\mathcal{T}, \mathcal{S}))$
  - $\mathcal{E} \leftarrow \mathcal{E} \setminus \overline{\Delta\mathcal{E}}$

Until $\mathcal{E} = \mathcal{E}_{old}$

Outline

Hybrid Algorithm (for Comparison)

- Partition representation: Block numbering function (non-interleaved)
- Transition representation: Relation (interleaved)
- Similar to generic signature-based splitting algorithm.

Hybrid Algorithm Signature Formula (First Try)

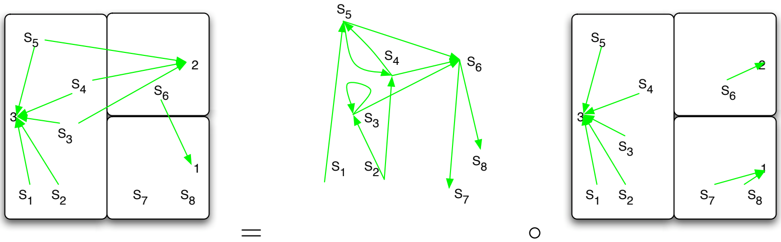
- S — State Space
- $P \subseteq S \times \mathbb{N}^{+}$ — Partition block number function of state
- $Q \subseteq \hat{S} \times \hat{S}$ — Transition relation
- $T \subseteq S \times \mathbb{N}^{+} \times \mathbb{N}^{+}$ — Signature map state to pairs of blocks
- $T(s) = \bigcup_{s' \in S} \{\langle P(s), P(s')\rangle | \langle s, s'\rangle \in Q\}$. (wrong)
- $T(s, b, b')$ iff $\exists s' \in S : (Q(s, s') \wedge P(s, b) \wedge P(s', b'))$.

Hybrid Algorithm Signature Formula

- S State Space
- $P \subseteq S \times [1, |S|]$ Partition block number function of state
- $Q \subseteq \hat{S} \times \hat{S}$ Transition relation
- $T \subseteq S \times [1, |S|] \times [0, |S|]$ Signature map to pairs of blocks
- $T(s) = \{\langle P(s), 0 \rangle\} \cup \bigcup_{s' \in S} \{\langle P(s), P(s') \rangle | \langle s, s' \rangle \in Q\}$.
- $T(s, b, b')$ iff $(P(s, b) \wedge b' = 0) \vee \exists s' \in S :$ $(Q(s, s') \wedge P(s, b) \wedge P(s', b'))$.

Hybrid Algorithm Signature

$$T = Q \circ P$$



Hybrid Algorithm signature Calculation

- State space MDD: $\mathcal{S}$
- Partition block number function MDD: $\mathcal{P} \subseteq S \times [1, |S|]$
- interleaved transition relation MDD: $\mathcal{Q} \subseteq \hat{S} \times \hat{S}$
- Signatures MDD: $\mathcal{T} \leftarrow \mathcal{W} \cup \mathcal{T}_{partial}$, where:
  - $\mathcal{W} = DC_3(\mathcal{P}, \{0\})$
  - $\mathcal{I} = [0, |\mathcal{S}|]$
  - $\mathcal{T}_{partial} = proj_{\vee 2}($ $DC_4(DC_3(\mathcal{Q}, \mathcal{I}), \mathcal{I}) \cap DC_4(DC_2(\mathcal{P}, \mathcal{S}), \mathcal{I}) \cap DC_1(DC_2(\mathcal{P}, \mathcal{I}), \mathcal{S})$ $)$
- $T(s, b, b')$ iff $(P(s, b) \wedge b' = 0) \vee \exists s' \in S :$ $(Q(s, s') \wedge P(s, b) \wedge P(s', b'))$.

Definitions for Extra Operators

- $DC_2(\mathcal{P},\mathcal{S}) \triangleq \underline{\mathcal{P}}$, where $\underline{\mathcal{P}}(x,y,z) = \mathcal{P}(x,z) \wedge \mathcal{S}(y)$
- $DC_3(\mathcal{Q},\mathcal{I}) \triangleq \underline{\mathcal{Q}}$, where $\underline{\mathcal{Q}}(x,y,z) = \mathcal{Q}(x,y) \wedge \mathcal{I}(z)$
- $DC_1(\mathcal{R},\mathcal{S}) \triangleq \underline{\mathcal{R}}$, where $\underline{\mathcal{R}}(x,y,z,h) = \mathcal{R}(y,z,h) \wedge \mathcal{S}(x)$
- $DC_4(\mathcal{R},\mathcal{I}) \triangleq \underline{\mathcal{R}}$, where $\underline{\mathcal{R}}(x,y,z,h) = \mathcal{R}(x,y,z) \wedge \mathcal{I}(h)$
- $proj_{\vee 2}(\mathcal{F}) \triangleq \mathcal{F}'$, where $\mathcal{F}'(x,y,z) = \bigvee c : \mathcal{F}(x,c,y,z)$
- Signatures MDD: $\mathcal{T} \leftarrow \mathcal{W} \cup \mathcal{T}_{partial}$, where:
  - $\mathcal{W} = DC_3(\mathcal{P},\{0\})$
  - $\mathcal{I} = [0,|\mathcal{S}|]$
  - $\mathcal{T}_{partial} = proj_{\vee 2}($ $DC_4(DC_3(\mathcal{Q},\mathcal{I}),\mathcal{I}) \cap DC_4(DC_2(\mathcal{P},\mathcal{S}),\mathcal{I}) \cap DC_1(DC_2(\mathcal{P},\mathcal{I}),\mathcal{S})$ $)$
- $\mathcal{W}(s,b,b')$ iff $\mathcal{P}(s,b) \wedge b' \in \{0\}$
- $T(s,b,b')$ iff $(P(s,b) \wedge b' = 0) \vee \exists s' \in S :$ $(Q(s,s') \wedge P(s,b) \wedge P(s',b'))$.

Substituting Extra Operators into $\mathcal{T}_{partial}$

- $DC_2(\mathcal{P},\mathcal{S}) \triangleq \underline{\mathcal{P}}$, where $\underline{\mathcal{P}}(x,y,z) = \mathcal{P}(x,z) \wedge \mathcal{S}(y)$
- $DC_3(\mathcal{Q},\mathcal{I}) \triangleq \underline{\mathcal{Q}}$, where $\underline{\mathcal{Q}}(x,y,z) = \mathcal{Q}(x,y) \wedge \mathcal{I}(z)$
- $DC_1(\mathcal{R},\mathcal{S}) \triangleq \underline{\mathcal{R}}$, where $\underline{\mathcal{R}}(x,y,z,h) = \mathcal{R}(y,z,h) \wedge \mathcal{S}(x)$
- $DC_4(\mathcal{R},\mathcal{I}) \triangleq \underline{\mathcal{R}}$, where $\underline{\mathcal{R}}(x,y,z,h) = \mathcal{R}(x,y,z) \wedge \mathcal{I}(h)$
- $proj_{\vee 2}(\mathcal{F}) \triangleq \mathcal{F}'$, where $\mathcal{F}'(x,y,z) = \bigvee c : \mathcal{F}(x,c,y,z)$
- $\mathcal{T}_{partial} = proj_{\vee 2}$
  - $DC_4(DC_3(\mathcal{Q},\mathcal{I}),\mathcal{I}) \cap DC_4(DC_2(\mathcal{P},\mathcal{S}),\mathcal{I}) \cap DC_1(DC_2(\mathcal{P},\mathcal{I}),\mathcal{S})$
- $\mathcal{T}_{partial}(s,b,b')$ iff $\bigvee s'$
  - $DC_4(DC_3(\mathcal{Q},\mathcal{I}),\mathcal{I})(s,s',b,b') \wedge$
    $DC_4(DC_2(\mathcal{P},\mathcal{S}),\mathcal{I})(s,s',b,b') \wedge$
    $DC_1(DC_2(\mathcal{P},\mathcal{I}),\mathcal{S})(s,s',b,b')$
- $T(s,b,b')$ iff $(P(s,b) \wedge b' = 0) \vee \exists s' \in S :$
  $(Q(s,s') \wedge P(s,b) \wedge P(s',b'))$.

Substituting Extra Operators into $\mathcal{T}_{partial}$

- $DC_2(\mathcal{P},\mathcal{S}) \triangleq \underline{\mathcal{P}}$, where $\underline{\mathcal{P}}(x,y,z) = \mathcal{P}(x,z) \wedge \mathcal{S}(y)$
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- $DC_4(\mathcal{R},\mathcal{I}) \triangleq \underline{\mathcal{R}}$, where $\underline{\mathcal{R}}(x,y,z,h) = \mathcal{R}(x,y,z) \wedge \mathcal{I}(h)$
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- $\mathcal{T}_{partial}(s,b,b')$ iff $\bigvee s'$
  - $[\mathcal{Q}(s,s') \wedge \mathcal{I}(b) \wedge \mathcal{I}(b')] \wedge [\mathcal{P}(s,b) \wedge \mathcal{S}(s') \wedge \mathcal{I}(b')] \wedge$ $[\mathcal{P}(s',b') \wedge \mathcal{I}(b) \wedge \mathcal{S}(s)]$
- $T(s,b,b')$ iff $(P(s,b) \wedge b' = 0) \vee \exists s' \in S :$ $(Q(s,s') \wedge P(s,b) \wedge P(s',b'))$.

Substituting Extra Operators into $\mathcal{T}_{partial}$

- $DC_2(\mathcal{P},\mathcal{S}) \triangleq \underline{\mathcal{P}}$, where $\underline{\mathcal{P}}(x,y,z) = \mathcal{P}(x,z) \wedge \mathcal{S}(y)$
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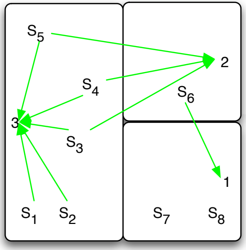
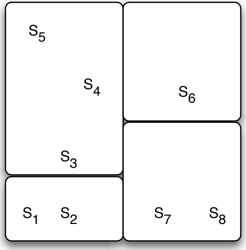
- $\mathcal{T}_{partial}(s,b,b')$ iff $\bigvee s'$
  - $[\mathcal{Q}(s,s')] \wedge [\mathcal{P}(s,b)] \wedge [\mathcal{P}(s',b')] \wedge \mathcal{S}(s) \wedge \mathcal{S}(s') \wedge \mathcal{I}(b) \wedge \mathcal{I}(b')$
- $T(s,b,b')$ iff $(P(s,b) \wedge b' = 0) \vee \exists s' \in S :$ $(Q(s,s') \wedge P(s,b) \wedge P(s',b'))$.

Hybrid Algorithm Block Splitting/Numbering

- S State Space
- $T \subseteq S \times [1, |S|] \times [0, |S|]$ Signature map to pairs of blocks
- $P' \subseteq S \times [1, |S|]$ Partition block number function of state
- New partition blocks for each different signature.
- Block number for each state according to its signature.
- $\exists f \in [1, |S|] \times [1, |S|] \times [0, |S|] : \forall s \in S : \forall b \in [1, |S|] :$ $P'(s, b)$ iff $\{\langle b_1, b_2 \rangle | f(b, b_1, b_2)\} = \{\langle b_1, b_2 \rangle | T(s, b_1, b_2)\}$.

Hybrid Algorithm Block Splitting

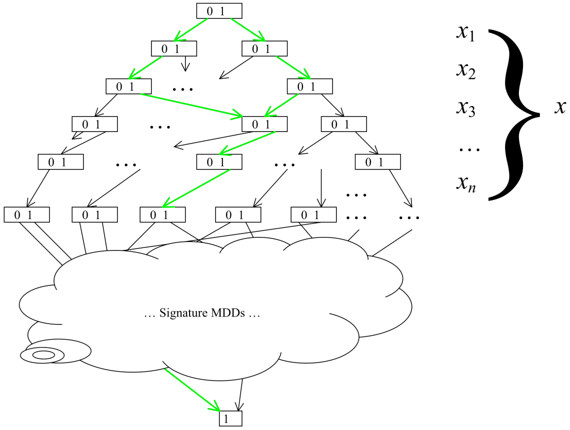
T: 

, P': 

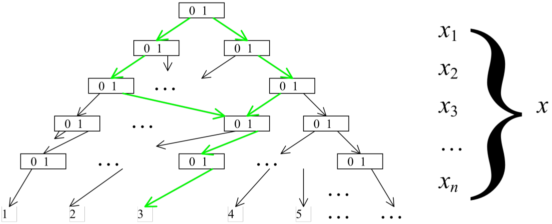
Hybrid Algorithm Block Renumbering Calculation

- Utilize canonicity of MDD
- Utilize fact that MDD is non-interleaved with state toward root
- Recursively DFS signature MDD $\mathcal{T}$
- Assign new partition number upon finding new signature.

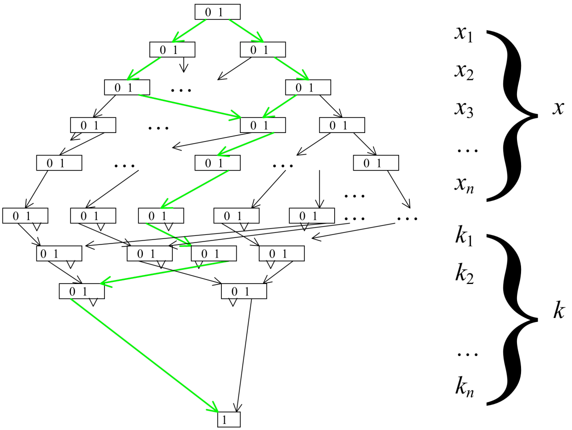
Hybrid Algorithm: Signatures MDD



Hybrid Algorithm: Block Renumbering $S \to \mathbb{N}$



Hybrid Algorithm: Block Renumbering $S \rightarrow \mathbb{N}$



Hybrid Algorithm Block Renumbering Algorithm

Assign new block number, corresponding to signature, to each state.

Algorithm: SigRenum(MDD $\mathcal{T}$)

Return SigRenum(MDD $\mathcal{T}$) from cache if possible.
If $\mathcal{T}$ is above signature level then

- let $\mathcal{R}$ = new MDD with each child $\mathcal{R}_i$ = SigRenum($\mathcal{T}_i$)

else

- let $\mathcal{R}$ = BDD for value of counter
- increment counter

Put $\mathcal{R}$ = SigRenum(MDD $\mathcal{T}$) into cache.
Return $\mathcal{R}$

Hybrid Algorithm

Given: Initial partition block numbering in variable $\mathcal{P}$, transition relation in $\mathcal{Q}$, state space in $\mathcal{S}$.
Returns final partition block numbering in $\mathcal{P}$.

Algorithm: refinement of block numbering using signature

Repeat:

- $\mathcal{P}_{old} \leftarrow \mathcal{P}$
- $\mathcal{T} \leftarrow \mathcal{W} \cup \mathcal{T}_{partial}$, where:
  - let: $\mathcal{W} \leftarrow DC_3(\mathcal{P}, \{0\})$, and: $\mathcal{I} \leftarrow [0, |\mathcal{S}|]$
  - $\mathcal{T}_{partial} \leftarrow proj_{\vee 2}$ $DC_4(DC_3(\mathcal{Q},\mathcal{I}),\mathcal{I}) \cap DC_4(DC_2(\mathcal{P},\mathcal{S}),\mathcal{I}) \cap DC_1(DC_2(\mathcal{P},\mathcal{I}),\mathcal{S})$
- $\mathcal{P} \leftarrow$ SigRenum($\mathcal{T}$)

Until $\mathcal{P} = \mathcal{P}_{old}$

Hybrid Algorithm with Transition Labeling

Given: Initial partition block numbering in variable $\mathcal{P}$, transition relation in $\mathcal{Q}$, state space in $\mathcal{S}$.
Returns final partition block numbering in $\mathcal{P}$.

Algorithm: refinement of block numbering using signature

Repeat:

- $\mathcal{P}_{old} \leftarrow \mathcal{P}$
- For each $t \in label$ loop:
  - $\mathcal{T} \leftarrow \mathcal{W} \cup \mathcal{T}_{partial}$, where:
    - let: $\mathcal{W} \leftarrow DC_3(\mathcal{P}, \{0\})$, and: $\mathcal{I} \leftarrow [0, |\mathcal{S}|]$
    - $\mathcal{T}_{partial} \leftarrow proj_{\vee 2}$ $DC_4(DC_3(\mathcal{Q}_t, \mathcal{I}), \mathcal{I}) \cap DC_4(DC_2(\mathcal{P}, \mathcal{S}), \mathcal{I}) \cap DC_1(DC_2(\mathcal{P}, \mathcal{I}), \mathcal{S})$
  - $\mathcal{P} \leftarrow$ SigRenum($\mathcal{T}$)

Until $\mathcal{P} = \mathcal{P}_{old}$

Example from Algorithm 1 Signature MDD

- Signatures MDD: $\mathcal{T} \leftarrow proj_{\vee 3}((DC_2(\mathcal{Q},\mathcal{S})) \cap (DC_1(\mathcal{E},\mathcal{S})))$
- Calculate: $((DC_2(\mathcal{Q},\mathcal{S})) \cap (DC_1(\mathcal{E},\mathcal{S})))$ using single recursive function.
- Avoid construction of intermediates: $DC_2(\mathcal{Q},\mathcal{S})$ and $DC_1(\mathcal{E},\mathcal{S})$.
- Recursive function will have 3 MDD parameters: $\mathcal{Q},\mathcal{E},\mathcal{S}$.
- Given $\mathcal{E} = \mathcal{E}^{-1}$ and $\mathcal{E} \subseteq S \times S$.
- Each recursive call level corresponds to level of output MDD.

Algorithm 6: Unprojected Relational Composition

Calculate: $\mathcal{R} = ((DC_2(\mathcal{Q}, \mathcal{S})) \cap (DC_1(\mathcal{E}, \mathcal{S})))$,
so that $\mathcal{R}(a, b, c)$ iff $\mathcal{Q}(a, c) \wedge \mathcal{E}(b, c) \wedge \mathcal{S}(a)$

Algorithm: UcompL(MDD $\mathcal{Q}, \mathcal{E}, \mathcal{S}$) (memoized)

- Leaf level: Return $\mathcal{Q} \cap \mathcal{E}$
- “a” level
  - Return new MDD $\mathcal{R}$ where child $\mathcal{R}_i =$ UcompL($\mathcal{Q}_i, \mathcal{E}, \mathcal{S}_i$)
- “b” level
  - Return new MDD $\mathcal{R}$ where child $\mathcal{R}_i =$ UcompL($\mathcal{Q}, \mathcal{E}_i, \mathcal{S}$)
- “c” level
  - Return new MDD $\mathcal{R}$ where child $\mathcal{R}_i =$ UcompL($\mathcal{Q}_i, \mathcal{E}_i, \mathcal{S}$)

Improvements Applied to Both Algorithms

- Improvement implemented as a single highly parameterized recursive function: GenericComposeQQ.
- Applied to: $\mathcal{T} \leftarrow proj_{\vee 3}((DC_2(\mathcal{Q},\mathcal{S})) \cap (DC_1(\mathcal{E},\mathcal{S}))),$ (signature for Algorithm 1).
- Applied to: $\mathcal{T}_{partial} = proj_{\vee 2}($ $DC_4(DC_3(\mathcal{Q},\mathcal{I}),\mathcal{I}) \cap DC_4(DC_2(\mathcal{P},\mathcal{S}),\mathcal{I}) \cap$ $DC_1(DC_2(\mathcal{P},\mathcal{I}),\mathcal{S})\ ),$ (signature for Hybrid Algorithm).
- Not applied to: $\overline{\Delta\mathcal{E}} \leftarrow proj_{\vee 3}(DC_2(\mathcal{T},\mathcal{S}) \dot\cup DC_1(\mathcal{T},\mathcal{S})),$ ($\mathcal{E}$ update for Algorithm 1).
- where $x \dot\cup y \triangleq (x \setminus y) \cup (y \setminus x)$
- Could have been (avoid calculating $(x \setminus y)$ and $(y \setminus x)$).

Outline

Transitive Closure (Finite $\hat{S}$)

Given: $t_{[\mathcal{E}]} \subseteq \hat{S} \times \hat{S}$ — indexed set of transition relations

Given: $S_{in} \subseteq \hat{S}$ — set of initial states

Returns: $S \subseteq \hat{S}$ — states reachable from S_{in} by transitions $t_{[\mathcal{E}]}$

Algorithm: *IterativeTransitiveClosure*$(t_{[\mathcal{E}]}, S_{in})$

- $S \leftarrow S_{in}$
- Repeat:
  - $S_{old} \leftarrow S$
  - For each $\alpha \in \mathcal{E}$ loop:
    - $S \leftarrow S \cup t_{[\alpha]}(S)$
- Until $S = S_{old}$
- Return: S

Saturation Transitive Closure (Finite $\hat{S}$)

Same givens and result as for previous Transitive Closure.

Algorithm: $SaturationClosure(t_{[\mathcal{E}]}, S_{in})$

- $S \leftarrow S_{in}$
- $S \leftarrow SaturateChildren(t_{[\mathcal{E}]}, S)$ ⋆
- Repeat:
  - $S_{old} \leftarrow S$
  - For each $\alpha \in \mathcal{E}$ loop:
  - - $S \leftarrow S \cup t_{[\alpha]}(S)$
      $S \leftarrow SaturateChildren(t_{[\mathcal{E}]}, S)$ ⋆
- Until $S = S_{old}$
- Return: S

Saturation Transitive Closure (Finite $\hat{S}$)

Same givens and result as for previous Transitive Closure.

Algorithm: $SaturationClosure(t_{[\mathcal{E}]}, S_{in})$

- $S \leftarrow S_{in}$
- $S \leftarrow SaturateChildren(t_{[\mathcal{E}]}, S)$ ⋆
- Repeat:
  - $S_{old} \leftarrow S$
  - For each $\alpha \in \mathcal{E}$ loop: If $Top(t_{[\alpha]})$ is top of S then: ⋆
  -
    - $S \leftarrow S \cup t_{[\alpha]}(S)$
      $S \leftarrow SaturateChildren(t_{[\mathcal{E}]}, S)$ ⋆
- Until $S = S_{old}$
- Return: S

Saturation Transitive Closure (Finite $\hat{S}$)

Same givens and result as for previous Transitive Closure.

Algorithm: $SaturationClosure(t_{[\mathcal{E}]}, S_{in})$

- $S \leftarrow S_{in}$
- $S \leftarrow SaturateChildren(t_{[\mathcal{E}]}, S)$ $\star$
- Repeat:
  - $S_{old} \leftarrow S$
  - For each $\alpha \in \mathcal{E}$ loop: If $Top(t_{[\alpha]})$ is top of S then: $\star$
  - - $S \leftarrow S \cup t_{[\alpha]}(S)$
      $S \leftarrow SaturateChildren(t_{[\mathcal{E}]}, S)$ $\star$
- Until $S = S_{old}$
- Return: S

Saturation Transitive Closure (Finite $\hat{S}$)

Same givens and result as for previous Transitive Closure.

Algorithm: $SaturationClosure(t_{[\mathcal{E}]}, S_{in})$

- $S \leftarrow S_{in}$
- $S \leftarrow SaturateChildren(t_{[\mathcal{E}]}, S)$ $\star$
- Repeat:
  - $S_{old} \leftarrow S$
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  -
    - $S \leftarrow S \cup t_{[\alpha]}(S)$
      $S \leftarrow SaturateChildren(t_{[\mathcal{E}]}, S)$ $\star$
- Until $S = S_{old}$
- Return: S

Helper Function for Saturation

Given: $t_{[\mathcal{E}]} \subseteq \hat{S} \times \hat{S}$ — indexed set of transition relations

Given: $S_{in} \subseteq \hat{S}$ — set of initial states

Returns: $S \subseteq \hat{S}$ — states reachable from S_{in} by transitions $t_{[\mathcal{E}]}$

where $Top(t_{[\alpha]})$ is below top of S

Algorithm: $SaturateChildren(t_{[\mathcal{E}]}, S_{in})$

- $S \leftarrow$ new MDD Where:
- child $S_{[i]} \leftarrow SaturationClosure(t_{[\mathcal{E}]}, S_{in[i]})$ $\forall i$
- Return: S

Saturation Transitive Closure (Finite $\hat{S}$)

Given: $t_{[\mathcal{E}]} \subseteq \hat{S} \times \hat{S}$ And: $S_{in} \subseteq \hat{S}$ Returns: $S \subseteq \hat{S}$

Algorithm: *SaturationClosure*$(t_{[\mathcal{E}]}, S_{in})$

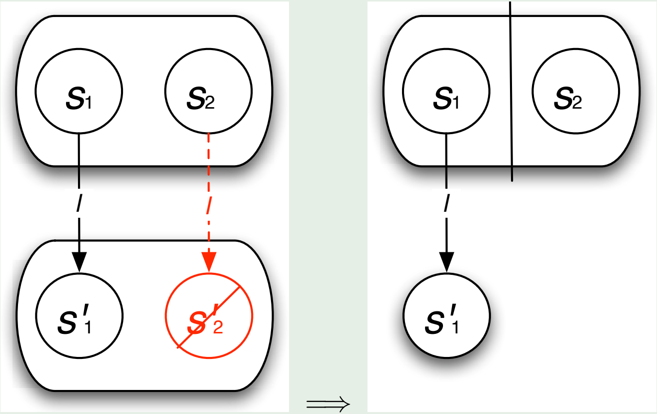
- $S \leftarrow S_{in}$
- $S_{[i]} \leftarrow \mathit{SaturationClosure}(t_{[\mathcal{E}]}, S_{[i]})$ $\forall i$
- Repeat:
  - $S_{old} \leftarrow S$
  - For each $\alpha \in \mathcal{E}$ loop:
  - For each $\alpha \in \mathcal{E}$ loop: If $\mathit{Top}(t_{[\alpha]})$ is top of S then:
  -
    - $S \leftarrow S \cup t_{[\alpha]}(S)$
    - $S_{[i]} \leftarrow \mathit{SaturationClosure}(t_{[\mathcal{E}]}, S_{[i]})$ $\forall i$
- Until $S = S_{old}$
- Return: S

Saturation Discussion.

- Child MDDs always Saturated
- Sharing Preserved
- Similar to local block iteration
-

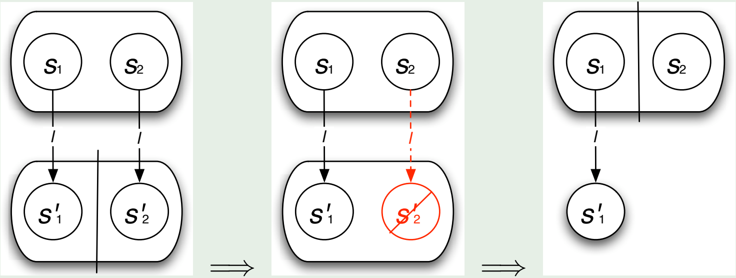
Splitting.

Example



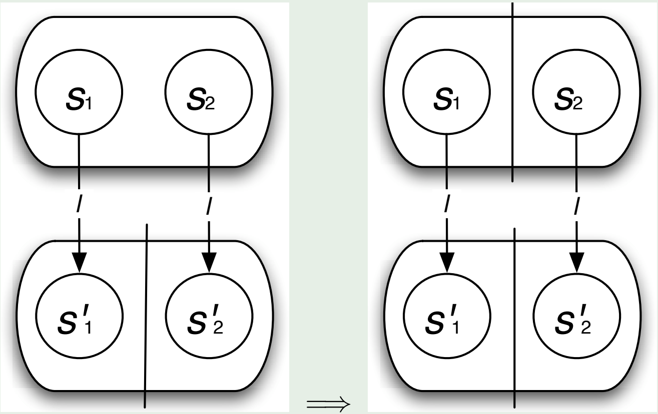
Splitting with Deterministic Transitions.

Example



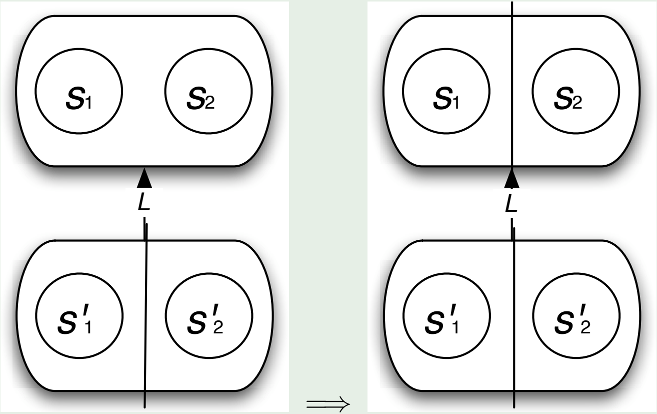
Splitting with Deterministic Transitions is a Transition.

Example



Splitting with Deterministic Transitions is a Transition.

Example



Splitting with Deterministic Transitions is a Transition.

- Given bisimulation problem:
- Transitions: $\mathcal{T}_{[\mathcal{E}]} \subseteq S \times S$
- Construct new domain: $\hat{\mathcal{B}} = S \times S$
- Create new transitions: $T_{[\mathcal{E}]} \subseteq \hat{\mathcal{B}} \times \hat{\mathcal{B}}$.
- $T_{[\alpha]}(\langle s_1, s_2 \rangle) =$ pairs $\langle s_3, s_4 \rangle$
- where $s_1 = \mathcal{T}_{[\alpha]}(s_3) \wedge s_2 = \mathcal{T}_{[\alpha]}(s_4)$ $\quad (\forall \alpha \in \mathcal{E})$
- $T_{[\alpha]} = (\mathcal{T}_{[\alpha]} \times \mathcal{T}_{[\alpha]})^{-1}$ $\quad (\forall \alpha \in \mathcal{E})$

Main Idea.

- Given bisimulation problem:
- Transitions: $\mathcal{T}_{[\mathcal{E}]} \subseteq S \times S$
- $T_{[\alpha]} = (\mathcal{T}_{[\alpha]} \times \mathcal{T}_{[\alpha]})^{-1}$ $(\forall \alpha \in \mathcal{E})$
- $\bar{\sim}$ is closed under $T_{[\mathcal{E}]}$
- Main Idea: Use Saturation to take closure of $T_{[\mathcal{E}]}$
- Then: $\sim = \hat{\mathcal{B}} \setminus \bar{\sim}$
- Initialization? closure applied to ?

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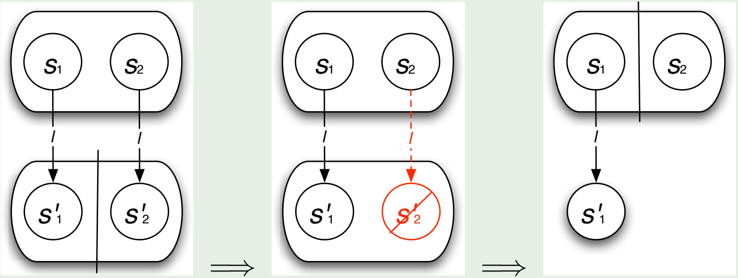
- $\overline{\sim}$ is closed under $T_{[\mathcal{E}]}$
- Main Idea: Use Saturation to take closure of $T_{[\mathcal{E}]}$
- Then: $\sim = \hat{\mathcal{B}} \setminus \overline{\sim}$
- Initialization? closure applied to ?

“Splitting” with Deterministic Transitions is Incomplete.

Example



Initial Set.

- Given bisimulation problem:
- Transitions: $\mathcal{T}_{[\mathcal{E}]} \subseteq S \times S$
- New domain: $\hat{\mathcal{B}} = S \times S$
- Initial Set: $\overline{\mathcal{B}}_{init} \subseteq \hat{\mathcal{B}}$, where only 1 member of each pair enables $\mathcal{T}_{[\alpha]}$, for some $\alpha \in \mathcal{E}$.
- Initial Set: $\overline{\mathcal{B}}_{init} = \bigcup_{\alpha \in \mathcal{E}} (\mathcal{S}_{[\alpha]} \times (S \setminus \mathcal{S}_{[\alpha]})) \cup ((S \setminus \mathcal{S}_{[\alpha]}) \times \mathcal{S}_{[\alpha]})$ where $\mathcal{S}_{[\alpha]} = \{s \in S | \exists s' : \langle s, s' \rangle \in \mathcal{T}_{[\alpha]}\}$.

Algorithm A

- Given bisimulation problem:
- Transitions: $\mathcal{T}_{[\mathcal{E}]} \subseteq S \times S$

Algorithm: $\overline{\textit{SaturationBisimulation}}(S, \mathcal{T}_{[\mathcal{E}]})$

- Define: $\hat{\mathcal{B}} = S \times S$
- For $(\alpha \in \mathcal{E})$ loop: $T_{[\alpha]} \leftarrow (\mathcal{T}_{[\alpha]} \times \mathcal{T}_{[\alpha]})^{-1}$
- Construct: $\overline{\mathcal{B}}_{init} \leftarrow \bigcup_{\alpha \in \mathcal{E}} (\mathcal{S}_{[\alpha]} \times (S \setminus \mathcal{S}_{[\alpha]})) \cup ((S \setminus \mathcal{S}_{[\alpha]}) \times \mathcal{S}_{[\alpha]})$ where $\mathcal{S}_{[\alpha]} = \{s \in S | \exists s' : \langle s, s' \rangle \in \mathcal{T}_{[\alpha]}\}$.
- $\overline{\approx} \leftarrow \textit{SaturationClosure}(T_{[\mathcal{E}]}, \overline{\mathcal{B}}_{init})$
- Return: $\hat{\mathcal{B}} \setminus \overline{\approx}$

$=\sim$

SMART Integration

- All code implemented in a single unit: “ms_lumping”.
- Invoked from SMART by a single C++ function call: “bigint ComputeNumEQClass(state_model *mdl);”
- Calculates largest bisimulation and returns number of equivalence classes.
- Invocation caused by “num_eqclass” function in model.
- Uses multiple caches supplied by SMART MDD library (Thanks, Min!).
- Uses operations: $\cup$, $\cap$, $\setminus$, new MDD, $||$, etc. from SMART MDD library.
- Implements operations for interleaved MDDs: $proj_{\vee}$, $\circ$, DC_{*}, $|classes|$
- Implements SigRenum

Summary of Our Bisimulation Algorithms

Three Algorithms:

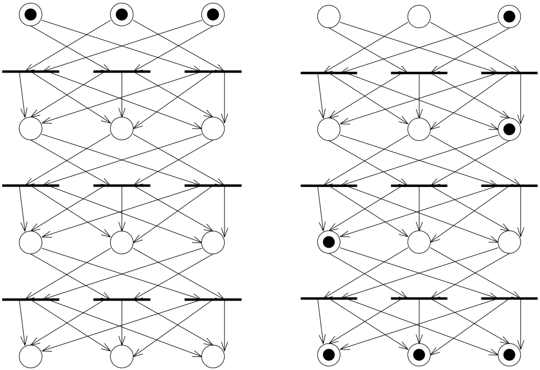
Fully Implicit	Hybrid	Saturation
Transition relation: Interleaved MDD	Transition relation: Interleaved MDD	Transition relation: Interleaved MDD
Partition: Equivalence, Interleaved MDD	Partition: Block number function MDD	Partition: Equivalence, Interleaved MDD
Method: Iterative Splitting	Method: Iterative Splitting	Method: Closure of Splitting Function

Dining Philosophers

Existing Petri net model, parameterized in number of philosophers N. Has $6N$ places and $4N$ transitions. Variable ordering/assignment to levels changed to avoid non-deterministic transitions. Ideal case for Interleaved Ordering

$3 \times N$ "Comb" and $N \times N$ "Comb"

Contrived Simple Petri net, parameterized in rows N and columns M. Has MN places and $M(N-1)$ transitions.



Summary of Our Models

Three Models:

Model:	N phil's	$3 \times N$ "Comb"	$N \times N$ "Comb"
Trans graph:	Cyclic	acyclic	acyclic
# places:	$O(N)$	$O(3N)$	$O(N^2)$
# transitions:	$O(N)$	$O(3N)$	$O(N^2)$
Token density:	$O(1)$	$O(1)$	$O(1)$
Depth	$O(N)$	$O(N)$	$O(N^2)$
Fanout S:	$O(1)$	$O(1)$	$O(1)$
Event span:	$O(1)$	$O(3)$	$O(N)$

Model Statistics

N philosophers

N	states	classes
2	18	17
3	76	76
4	322	321
5	1364	1363
6	5778	5777
7	2.4×10^4	2.4×10^4
8	1.0×10^5	1.0×10^5
9	4.3×10^5	4.3×10^5
10	1.9×10^6	1.9×10^6
11	7.9×10^6	7.9×10^6
12	3.3×10^7	3.3×10^7
13	1.4×10^8	1.4×10^8
14	6.0×10^8	6.0×10^8

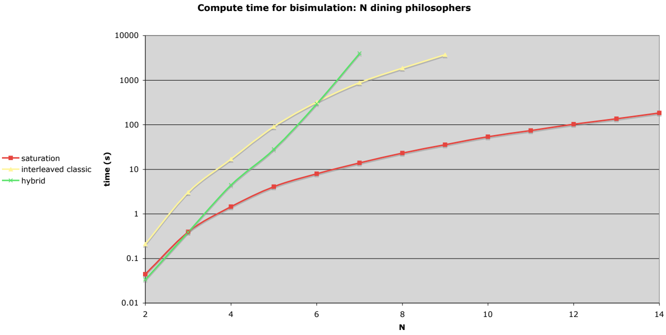
$3 \times N$ comb

N	states	classes
2	4	2
3	13	3
4	40	4
5	121	5
6	364	6
7	1093	7
8	3280	8
9	9841	9
10	3.0×10^4	10
11	8.9×10^4	11
12	2.7×10^5	12
13	8.0×10^5	13
14	2.4×10^6	14
15	7.2×10^6	15
16	2.2×10^7	16
17	6.5×10^7	17
18	1.9×10^8	18
19	5.8×10^8	19
20	1.7×10^9	20

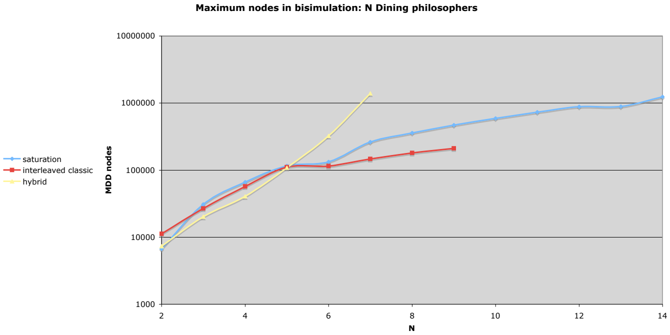
$N \times N$ comb

N	states	classes
2	4	4
3	13	3
4	85	4
5	781	5
6	9331	6
7	1.4×10^5	7
8	2.4×10^6	8
9	4.8×10^7	9
10	1.1×10^9	10
11	2.9×10^{10}	11
12	8.1×10^{11}	12
13	2.5×10^{13}	13
14	8.5×10^{14}	14
15	3.1×10^{16}	15
16	1.2×10^{18}	16
17	5.2×10^{19}	17
18	2.3×10^{21}	18
19	1.1×10^{23}	19
20	5.5×10^{24}	20

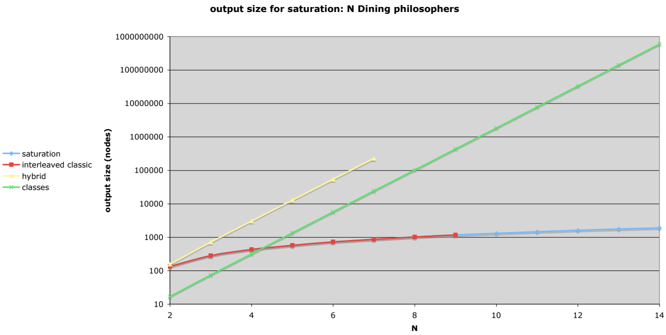
Run-Time for Dining Philosophers



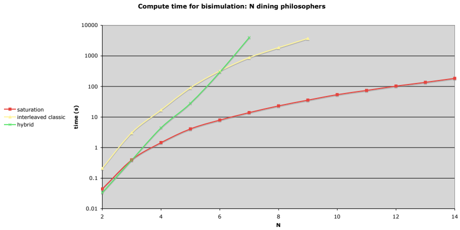
Space for Dining Philosophers

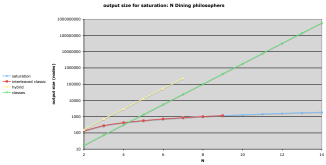


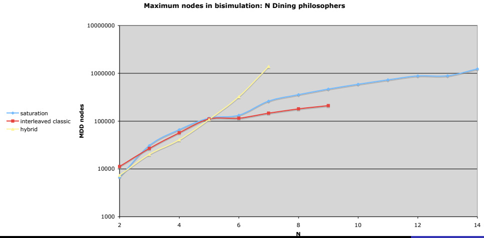
Output Size for Dining Philosophers



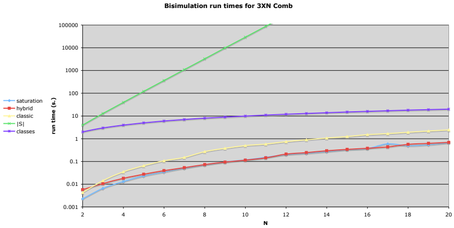
Combined Dining Philosophers Results

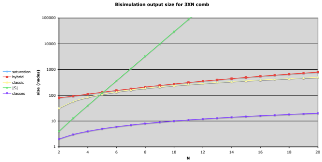


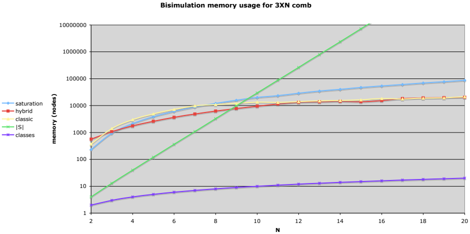




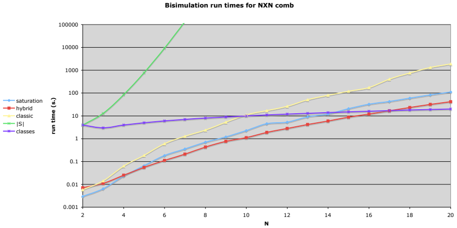
Combined 3 × N “Comb” Results

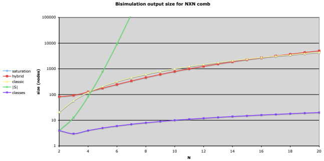


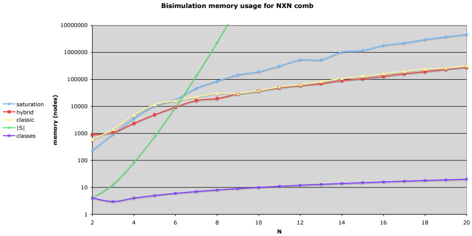




Combined $N \times N$ “Comb” Results







Discussion of Results

Qualitative evaluation of Quantitative results:

- Saturation performed well in all cases (especially D. P.).
- Classic algorithm had surprisingly better memory use.
- Saturation was not always fastest.

Additional Thoughts:

- This is approximately what we sought.
- Additional optimizations are possible.
- Hybrid algorithm is not exactly the same as fastest known.

Future Work

Improvements to current work:

- Extend to non-deterministic transitions.
- Additional models.
- Increase operator integration.
- Quantification/projection improvements.
- “Weak” bisimulation (invisible transitions).

Other related work:

- Implement fastest (previously) known algorithm.
- SMART library improvements.
- If possible, apply to lumping problem.

Summary

- Implementation of three bisimulation algorithms in SMART
- Comparison using three Petri net models.
- Obtained algorithm with good performance and (relatively) small output

- Future:
  - Improve and extend to non-deterministic transitions.
  - Compare with fastest (previously) known algorithm.
  - Publish.

The End

fin

After The End

(Click here for a reference.)

Ways to Represent Partitions

1. Equivalence Relation (Non-Interleaved) (link)
   - $\langle s_1, s_2 \rangle \mid s_1, s_2 \in S$
   - Variable ordering: $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$
2. Equivalence Relation (Interleaved) (link)
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   - Variable ordering: $x_1, y_1, x_2, y_2, x_3, y_3, \ldots$
3. Lists of Partition Blocks (link)
   - $B_1, B_2, B_3, B_4, \ldots \mid B_* \subseteq S$
   - Variable ordering: $x_1, x_2, x_3, \ldots$
4. Block Numbering/function of state (link)
   - $\langle s, n \rangle \mid s \in S, n \in \mathbb{N}$
   - Variable ordering: $x_1, x_2, x_3, \ldots k_1, k_2, k_3, \ldots$

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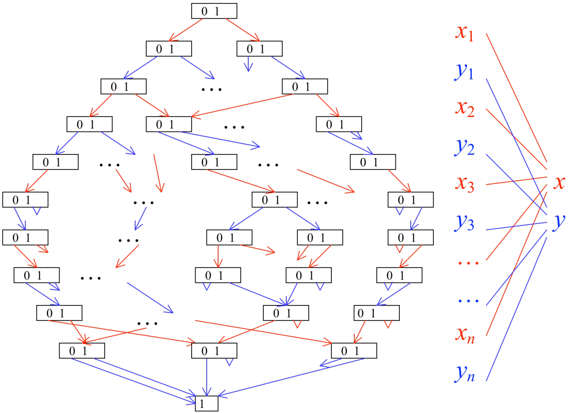
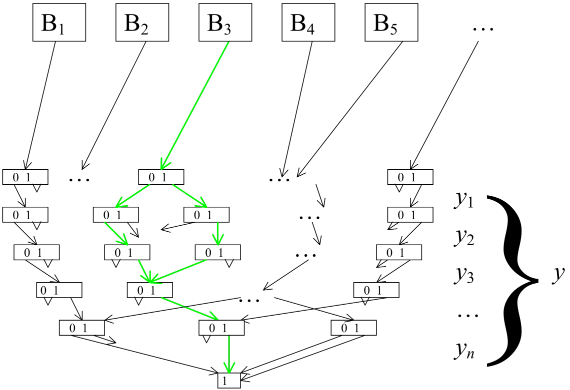
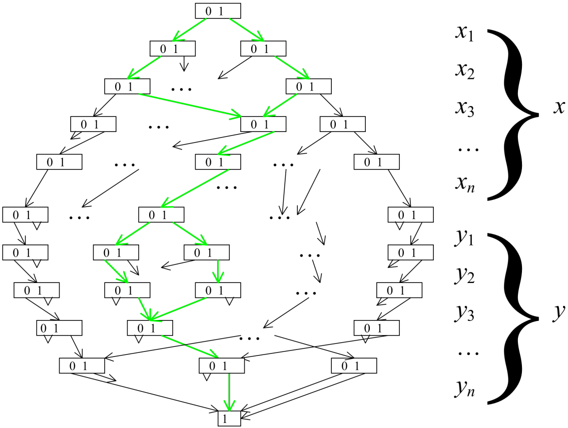
Partition Representation: Equivalence Relation (Interleaved) $\{\langle x, y\rangle | E(x, y)\}$



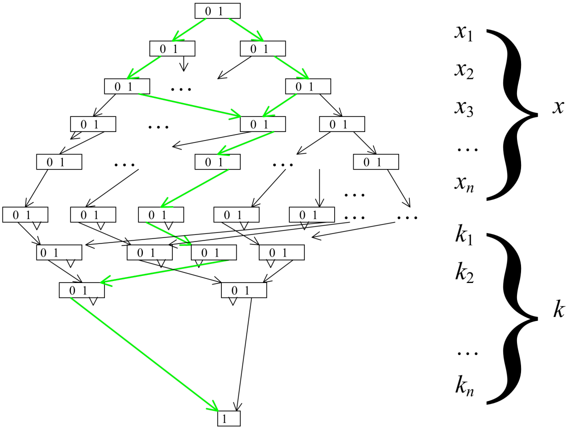
Partition Representation: Lists of Partition Blocks (or Array etc) $\mathbb{N} \rightarrow S$



Partition Representation: Equivalence Relation (Non-Interleaved) $\{\langle x, y\rangle | E(x, y)\}$



Partition Representation: Block Numbering/function of state $S \rightarrow \mathbb{N}$



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